A ring. $S \subseteq A$ m.c. set

$\Rightarrow S^{-1}A = \{\frac{a}{s} : s \in S, a \in A\}$ where $\frac{a}{s} = \frac{a'}{s'} \Leftrightarrow \exists t \in S : as't = a'st$

$\exists$ ring hom. $j = j_S : A \to S^{-1}A,\ a \mapsto \frac{a}{1}$

Note: $\frac{s}{1} \cdot \frac{1}{s} = \frac{1}{1} \Rightarrow j(S) \subseteq (S^{-1}A)^{\times}$

Universal Property of $(S^{-1}A, j)$: For every ring hom. $\varphi : A \to B$ with $\varphi(S) \subseteq B^{\times}$, there exists a unique ring hom $\overline{\varphi} : S^{-1}A \to B$ s.t. $\varphi = \overline{\varphi} \circ j$

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "\varphi"'] & S^{-1}A \arrow[d, dashed, "\exists!\overline{\varphi}"] \arrow[dl, phantom, "="] \\ & B \end{tikzcd}$$

Proof Sketch: **Uniqueness:** $\underbrace{\overline{\varphi}\left(\tfrac{s}{1}\right)}_{=\varphi(s)} \cdot \overline{\varphi}\left(\tfrac{a}{s}\right) = \overline{\varphi}\left(\tfrac{a}{1}\right) \cancel{\overline{\varphi}\left(\tfrac{1}{s}\right) \cdot \overline{\varphi}\left(\tfrac{s}{1}\right)} = \varphi(a)$

$\overset{\varphi(s) \in B^{\times}}{\Rightarrow} \overline{\varphi}\left(\frac{a}{s}\right) = \varphi(a)\varphi(s)^{-1} \quad (*)$

Existence: Check $(*)$ is well-defined. (Easy Exc) $\square$

Note: $\varphi(a)\varphi(s)^{-1} = 0 \Leftrightarrow \varphi(a) = 0$, so

$\operatorname{Ker}(\overline{\varphi}) = \{\frac{a}{s} : a \in \operatorname{Ker}\varphi, s \in S\}$

⚠ $j(a) = 0 \Leftrightarrow \frac{a}{1} = 0 \Leftrightarrow \exists s \in S : \underline{a \cdot s} = 0 \cdot s = \underline{0}$

$\Rightarrow \operatorname{Ker}(j) = \{a \in A : \exists s \in S : as = 0\}$ can be $\neq 0$!

Def: If $S = \{\text{non-zero-divisors of } A\}$, then $S^{-1}A =: \mathcal{F}(A)$ is the total ring of fractions (or total quotient ring) of A.

If A is a domain: field of fractions = fraction field = quotient field.

total ring of fractions

If A is a domain: field of fractions = fraction field = quotient field.

If S contains no zero-divisors, then $S^{-1}A \hookrightarrow \mathcal{F}(A)$ canonically (by UP).

Def: • If $S = A \setminus P$ for $P \in \operatorname{Spec}(A)$, write A_P

• If $S = \{1, a, a^2, \dots\}$ for $a \in S$, write A_a

Exm: p prime number, $\mathbb{Z}_{(p)} \cong \{ \frac{a}{b} \in \mathbb{Q} : p \nmid b \}$

$$\mathbb{Z}_p = \left\{ \frac{a}{p^k} \in \mathbb{Q} : a \in \mathbb{Z},\ k \geq 0 \right\}$$

Recall: An A-algebra is a ring B that is simultaneously an A-module (with same addition and $a \cdot (b_1 b_2) = (a \cdot b_1) b_2 = b_1 (a \cdot b_2)$) equivalently it is a ring hom. $f : A \to B$.

[„$\Rightarrow$" $a \mapsto a \cdot 1_B$ is a ring hom, „$\Leftarrow$" $a \cdot b := f(a) b$ is an A-module structure]

Exm: $A[x]$ is an A-algebra; Rings $\hat{=}$ $\mathbb{Z}$-algebras;

If $I \trianglelefteq A$, $A \to A/I$ is an A-algebra.

If $S \subseteq A$ is mc set $\Rightarrow j : A \to S^{-1}A$ is an A-algebra.

If $f : A \to B$ is an A-algebra, we can

• extend ideals: $I \trianglelefteq A \longmapsto f(I)B := \left\{ \sum_{i=1}^{n} f(x_i) b_i : x_i \in I, b_i \in B \right\} \trianglelefteq B$

• contract ideals: $J \trianglelefteq B \longmapsto f^{-1}(J) \trianglelefteq A$ [If $A \subseteq B$, f inclusion: $f^{-1}(J) = J \cap A$]

Notation: For $j : A \to S^{-1}A$ localization, $I \trianglelefteq A$ one writes $I \cdot S^{-1}A := j(I) S^{-1}A \overset{\text{check}}{=} \{ \frac{a}{s} : a \in I, s \in S \}$

Def: For $I \trianglelefteq A$, $X \subseteq A$, let $(I : X) := \{ a \in A : aX \subseteq I \}$ („colon ideal")

Note: $(I : X) \trianglelefteq A$.

Prop 2.1: Let $S \subseteq A$ be an mc set, $j : A \to S^{-1}A$ localization

(1) $\forall J \trianglelefteq S^{-1}A$, $J = j(j^{-1}(J)) \cdot S^{-1}A$

(1) $\forall J \trianglelefteq S^{-1}A, \quad J = j(j^{-1}(J)) \cdot SA$

In particular: Every ideal of $S^{-1}A$ is an extension of an ideal of A

(2) $\forall I \trianglelefteq A: \quad j^{-1}(j(I)\cdot S^{-1}A) = j^{-1}(I \cdot S^{-1}A) \overset{(*)}{=} \bigcup_{s\in S} (I:s) \overset{(**)}{\supseteq} I$

(3) $I \cdot S^{-1}A = S^{-1}A \iff I \cap S \neq \emptyset$.

<u>Proof:</u> (1) „$\supseteq$" ✓ „$\subseteq$" Let $\frac{a}{s} \in J$ $(a \in A, s \in S) \Rightarrow \underbrace{\frac{a}{1}}_{=j(a)} \in J$

$\Rightarrow a \in j^{-1}(J) \Rightarrow \frac{a}{1} \in j(j^{-1}(J)) \Rightarrow \frac{a}{s} \in \langle j(j^{-1}(J))\rangle_{S^{-1}A} = j(j^{-1}(J))\, S^{-1}A$

(2) $(**)$: Because $1 \in S$ and $I = (I:1)$

$(*)$ $a \in j^{-1}(I \cdot S^{-1}A) \iff \frac{a}{1} = \frac{x}{s}$ with $x \in I, s \in S$

$\iff \exists x \in I, s,t \in S: \quad ast = \underbrace{xt}_{\in I}$

$\iff \exists s' \in S: \quad a \in (I:s')$

(3) „$\Leftarrow$" because $j(S) \subseteq (S^{-1}A)^\times$

„$\Rightarrow$": Then $\frac{1}{1} = \frac{x}{s}$ with $x \in I, s \in S \Rightarrow \exists t: st = xt \in I \cap S$. $\square$

<u>Cor 2.2</u> If $P \in \mathrm{Spec}(A)$, then A_p is a local ring, with unique maximal ideal PA_p.

<u>Proof:</u> By (1),(3) and $S = A \setminus P$. $\square$

<u>Exm:</u> ·) $\mathbb{Z}_{(p)}$ w. p prime is local w. max ideal $\left\{\frac{a}{b} \in \mathbb{Q} : p \nmid b, p \mid a\right\}$

$\mathbb{Z}_{(0)} = \mathbb{Q}$ (local w. max ideal $\underline{0}$)

·) K field, $P = (x,y) \trianglelefteq K[x,y] \Rightarrow$

$K[x,y]_{(x,y)} = \left\{\frac{f}{g} : f,g \in K[x,y],\ g(0,0) \neq 0\right\}$ local,

max ideal $(x,y)K[x,y]_{(x,y)} = \left\{\frac{f}{g} : f,g \in K[x,y]: \ g(0,0) \neq 0,\ f(0,0) = 0\right\}$.

max ideal $(x,y)\,k[x,y]_{(x,y)} = \{\frac{f}{g} : f,g \in k[x,y] : g(0,0)\neq 0, f(0,0)=0\}$.

⚠ Different ideals can localize to the same one (even if non-trivial), e.g. in $\mathbb{Z}_{(2)}$: $2\mathbb{Z}_{(2)} = 6\mathbb{Z}_{(2)}$, because $3\in\mathbb{Z}_{(2)}^{\times}$.

Cor 2.3 There is a bijection

$$\begin{array}{ccc} \{P\in \operatorname{Spec}(A) : S\cap P=\emptyset\} & \longleftrightarrow & \operatorname{Spec}(S^{-1}A) \\ P & \longmapsto & P\cdot S^{-1}A \\ j^{-1}(Q) & \longleftarrow\!\shortmid & Q \end{array}$$

Proof: Let $j: A\to S^{-1}A$.

- Let $Q\in \operatorname{Spec}(S^{-1}A) \Rightarrow j^{-1}(Q)\in \operatorname{Spec}(A)$ [preimages of prime ideals under ring hom. are prime ideals]

  By C2.1: $j^{-1}(Q)\cdot S^{-1}A = Q$ and $j^{-1}(Q)\cap S=\emptyset$.

- Let $P\in\operatorname{Spec}(A)$, $S\cap P=\emptyset$

  $P\cdot S^{-1}A$ is prime: $P\cdot S^{-1}A = \{\frac{x}{s} : x\in P, s\in S\}$

  Suppose $\frac{a}{s}\cdot\frac{b}{t}\in P\cdot S^{-1}A \Rightarrow \frac{ab}{st} = \frac{x}{s'}$ w. $x\in P$, $s'\in S$

  $\exists u\in S$:
  $\Rightarrow abs'u = xstu \in P \overset{s'u\notin P}{\Rightarrow} ab\in P \Rightarrow a\in P$ or $b\in P$.

  Now: $j^{-1}(P\cdot S^{-1}A) = \bigcup_{s\in S} \underbrace{(P:s)}_{=P \text{ because } s\notin P} = P$. $[xs\in P\Rightarrow x\in P]$ □

Cor 2.4 (1) $\mathcal{N}(A) = \bigcap_{P\in\operatorname{Spec}(A)} P$ [compare: $\mathcal{J}(A) = \bigcap_{M\in\operatorname{Max}(A)} M$.]

(2) If $I\trianglelefteq A$, then $\sqrt{I} = \bigcap_{\substack{P\in\operatorname{Spec}(A)\\ I\subseteq P}} P$

Proof: (1) „$\subseteq$": Let $P\in\operatorname{Spec}(A)$, $a\in\mathcal{N}(A)$.

$\Rightarrow \exists n\geq 1$: $a^n = 0\in P \Longrightarrow a\in P$

„$\supseteq$": Let $a\in\bigcap P \overset{C2.3}{\Longrightarrow} A$ is a ring with $\operatorname{Spec}(A)=\emptyset$

$\exists n \geq 1: a^n = 0 \in P \implies a \in P$

"$\supseteq$": let $a \in \bigcap_{P \in \operatorname{Spec}(A)} P \overset{C2.3}{\implies} A_a$ is a ring with $\operatorname{Spec}(A) = \emptyset$

$\implies A_a = \underline{0}$ (otherwise $\exists$ max. ideals)

$\implies 1 \in \ker(j: A \to A_a) \xRightarrow{S = \{1, a, a^2, \dots\}} \exists n \geq 0: \quad a^n \cdot 1 = 0 \implies a \in \mathcal{N}(A)$

(2) Apply (1) to A/I, noting

$$\mathcal{N}(A/I) = \sqrt{I}/I. \qquad \square$$

2.1 Localization of modules

Let $M \in A\text{-Mod}$.

$$(m, s) \sim (m', s') \;:\Longleftrightarrow\; \exists t \in S: \quad m s' t = m' s t$$

is an equiv. relation on $M \times S$, $\frac{m}{s} := [(m, s)]_\sim$

$\implies S^{-1}M := \{\frac{m}{s} : m \in M, s \in S\}$ is an $S^{-1}A$-module, via

$$\frac{a}{s} \cdot \frac{m}{s'} := \frac{am}{ss'}$$

There is an A-hom $j: M \to S^{-1}M$, $m \mapsto \frac{m}{1}$.

$$\operatorname{Ker}(j) = \{m \in M : \exists s \in S : sm = 0\}.$$

If $f: M \to N$ is a hom of A-modules, then

$$S^{-1}f: S^{-1}M \to S^{-1}N, \quad \frac{m}{s} \mapsto \frac{f(m)}{s} \quad \text{is an } S^{-1}A\text{-hom.}$$

$\implies$ Localization is a functor $A\text{-}\underline{\text{Mod}} \to S^{-1}A\text{-}\underline{\text{Mod}}$.

Prop 2.5 Localization is an exact functor. I.e., if

$M \xrightarrow{f} N \xrightarrow{g} P$ is exact in A-Mod, then

$S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}P$ is exact (in $S^{-1}A$-Mod)

$S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}P$ is exact (in $S^{-1}A$-Mod)

<u>Proof</u>: $g\circ f \Rightarrow S^{-1}(f\circ g) = S^{-1}f \circ S^{-1}g = 0 \Rightarrow \mathrm{Im}(S^{-1}f) \subseteq \mathrm{Ker}(S^{-1}g)$.

<u>$\mathrm{Ker}(S^{-1}g) \subseteq \mathrm{Im}(S^{-1}f)$</u>: Let $\frac{n}{s} \in \mathrm{Ker}(S^{-1}g) \Rightarrow \frac{g(n)}{s} = 0$

$\Rightarrow \exists t \in S: \quad 0 = g(n)t = g(nt) \Rightarrow nt \in \mathrm{Ker}(g) = \mathrm{Im}(f)$.

Let $nt = f(m)$ with $m \in M$.

$\underset{(\text{in } S^{-1}N)}{\Rightarrow} \frac{n}{s} = \frac{nt}{st} = \frac{f(m)}{st} = S^{-1}f\left(\frac{m}{st}\right)$. $\square$

So if $M \leq N$, then wlog $S^{-1}M \leq S^{-1}N$,

$S^{-1}M = \{\frac{m}{s} \in S^{-1}N : \ s \in S, m \in M\}$

<u>In particular</u>: If $I \trianglelefteq A$, then $S^{-1}I = I \cdot S^{-1}A \trianglelefteq S^{-1}A$

<u>Cor 2.6</u> Let $N \in A$-Mod, $M, M' \leq N$

(1) $S^{-1}(M+M') = S^{-1}M + S^{-1}M'$

(2) $S^{-1}(M \cap M') = S^{-1}M \cap S^{-1}M'$

(3) $S^{-1}(N/M) \cong S^{-1}N / S^{-1}M$ as $S^{-1}A$-modules

<u>Proof</u>: (1) ✓

(2) „$\subseteq$" ✓, „$\supseteq$" Let $\frac{m}{s} = \frac{m'}{t}$ with $s,t \in S$, $m \in M$, $m' \in M'$

$\Rightarrow \exists u \in S: \quad mtu = m'su \in M \cap M'$

$\Rightarrow \frac{m}{s} = \frac{mtu}{stu} \in S^{-1}(M \cap M')$.

(3) By exactness, $0 \to M \hookrightarrow N \longrightarrow N/M \to 0$ induces the SES

$0 \to S^{-1}M \to S^{-1}N \to S^{-1}(N/M) \to 0 \Rightarrow S^{-1}(N/M) \cong S^{-1}N / S^{-1}M$. $\square$

<u>Remark</u>: If $I \trianglelefteq A$, then $S^{-1}\sqrt{I} = \sqrt{S^{-1}I} \trianglelefteq S^{-1}A$, in particular $S^{-1}\mathcal{N}(A) = \mathcal{N}(S^{-1}A)$ (Nice Exercise, a bit like (2))

<u>Lemma 2.7</u> Let $I \trianglelefteq A$, $\pi: A \to A/I$ canonical epi, $S \subseteq A$ mc set.

Lemma 2.7 Let $I \trianglelefteq A$, $\pi: A \to A/I$ canonical epi, $S \subseteq A$ mc set.

($\Rightarrow T := \pi(S) \subseteq A/I$ is an mc set).

If $M \in A\text{-Mod}$ with $IM = 0$, then M is an A/I-module, and

$$S^{-1}M \cong T^{-1}M \qquad \text{(canonically)}$$

Proof: $\varphi: S^{-1}M \longrightarrow T^{-1}M$, $\frac{m}{s} \longmapsto \frac{m}{\pi(s)}$ is clearly an A-module epi.

Check injectivity: Suppose $\frac{m}{\pi(s)} = \frac{0}{1}$ in $T^{-1}M$ $\Rightarrow \exists t \in T$: $mt = 0$

$t = \pi(s')$, $s' \in S$ $\Rightarrow$ $0 = m\pi(s') = m(s'+I)$ $\Rightarrow$ $ms' \in MI = 0$. $\square$